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A COMBINATION THEOREM FOR RELATIVELY HYPERBOLIC GROUPS

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Abstract

Given a graph of δ-hyperbolic spaces, this paper gives sufficient conditions that ensure that the graph itself is δ-hyperbolic. As an application, a simple proof is given to show that limit groups are relatively hyperbolic.

1. *Introduction*

In his work on Diophantine equations over free groups, Sela introduced limit groups. He showed that this class of groups coincides with the class of ω-residually free groups, which had already been extensively studied. One of the most important results is a structure theorem for limit groups, given by Kharlampovich and Myasnikov [**7**, **8**] and Sela [**10**].

This paper introduces a number of interesting questions that one might ask about limit groups. We were interested in describing the set of homomorphisms from an arbitrary finitely generated group G into a limit group L, namely $\mathrm{Hom}(G, L)$. A key tool in studying $\mathrm{Hom}(G, L)$ is a δ-hyperbolic space on which the given limit group L acts freely, by isometries. We construct such a space in Section 3. That the space we constructed is δ-hyperbolic follows from Theorem 2.3, which gives the conditions that a graph of hyperbolic spaces has to satisfy in order to be a hyperbolic space itself. The proof of this theorem is given in Section 2, and is an adaptation of the proof of the Bestvina–Feighn combination theorem [**1**] to a different setting.

The existence of such spaces for limit groups gives an answer to the question of whether limit groups are hyperbolic relative to their maximal noncyclic abelian subgroups. This question was answered affirmatively by Dahmani [**5**], who proved a combination theorem for geometrically finite convergence groups, using different methods.

2. *A combination theorem*

Let X be a connected cell complex which is a graph of spaces such that there is a map $p: X \longrightarrow \Gamma$ onto a finite graph Γ. Let X_e denote the preimage of the midpoint of an edge e in Γ, and let X_v denote the preimage of a component of $\Gamma \setminus \{\text{midpoints of all edges}\}$ that contains the vertex v. We require that X_e and X_v be connected, and that their inclusions into X induce inclusions on fundamental groups. There is an induced map $\tilde{p}: \tilde{X} \longrightarrow T$ from the universal cover of X onto a $\pi_1(X)$-tree T such that $T/\pi_1(X)$ is isomorphic to Γ. If we assign a length 1 to each edge of X, we obtain an induced combinatorial metric on the 1-skeleton of $\tilde{X}$, which can then be extended to a metric on $\tilde{X}$.

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We say that X is a *graph of negatively curved spaces* if every vertex space X_v is negatively curved. As a reminder, we note that a cell complex X (not necessarily finite) is said to be *negatively curved* if there exists a constant $A = A(X)$ such that each inessential circuit bounds a disk of combinatorial area that is bounded above by A times the combinatorial length of the circuit.

We know that in δ-hyperbolic spaces, geodesic triangles are δ-thin. We have a similar fact for polygons; in fact, for quasigeodesic polygons. The following proposition can be found in [**6**] and [**1**].

Proposition 2.1. *Let Z be a δ-hyperbolic space, and let $\tau \geqslant 1$ be a constant. There exist a function $B(x) = O(\log x)$ and a linear function $C(x)$, each depending only on Z and τ, with the following property. If $\Delta : D^2 \longrightarrow Z$ is a disk with boundary a k-sided τ-quasigeodesic polygon, then there exist a finite tree S and a map $r : D^2 \longrightarrow S$ such that the following statements hold.*

(i) *The number of valence-one vertices of S is k.*
(ii) *For a and b in S^1, we have $d_Z(\Delta(a), \Delta(b)) \leqslant d_S(r(a), r(b)) + B(k)$.*
(iii) *$r^{-1}(s)$ is a properly embedded finite tree in D^2 for $s \in S$.*
(iv) *If E is an edge of S, then r restricted to $r^{-1}(\mathrm{Interior}(E))$ is an I-bundle.*
(v) *For a_1, b_1 on the same side of the polygon and a_2, b_2 on the same side of the polygon such that $r(a_1) = r(a_2) \in E$ and $r(b_1) = r(b_2) \in E$, we have*

$$\begin{aligned}\ell(\Delta(&\textit{the circular arc } a_1b_1 \textit{ in the edge of the polygon}))\\ &\leqslant C(\ell(\Delta(\textit{the circular arc } a_2b_2 \textit{ in the edge of the polygon}))).\end{aligned}$$

Such a map r is called a *resolution* of the quasigeodesic polygon. A *singular fiber* of the resolution is a fiber that is not isomorphic to I.

We will say that a graph of spaces X is *partially qi-embedded* if every edge space $\widetilde{X_e}$ is quasiisometrically embedded in at least one of the vertex spaces $\widetilde{X_v}$ and $\widetilde{X_w}$, where v and w are endpoints of the edge e in Γ. We further ask that all qi-constants be equal (we can do this, since Γ is a finite graph).

A graph of spaces X is *qi-consistent* if the following holds: if one of the edge spaces adjacent to a vertex space $\widetilde{X_v}$ qi-embeds into it, then the same is true for all adjacent edge spaces. We will call such vertex spaces *good*.

A graph of spaces X is called *tight* if, whenever the distinct lifts $\widetilde{X_e}$ and $\widetilde{X_f}$ of edge spaces qi-embed into the same vertex space $\widetilde{X_v}$, then the intersection of any of their Hausdorff neighborhoods in $\widetilde{X_v}$ is a compact set. Note that we allow $e = f$.

Remark 2.2. If the graph of spaces is tight, then we have the following. Suppose that $\widetilde{X_e}$ and $\widetilde{X_f}$ qi-embed into the same vertex space $\widetilde{X_v}$. If we fix a constant $k < \infty$, then

$$L = \max\{\ell(S_1) : \exists S_2 \subset \widetilde{X_f} \text{ such that } d_H(S_1, S_2) \leqslant k\}$$

is finite, where S_1 and S_2 are quasigeodesics in $\widetilde{X_e}$ and $\widetilde{X_f}$, respectively. Note that L depends on the qi-constants for S_1 and S_2.

Theorem 2.3. *If a graph of negatively curved spaces X is partially qi-embedded, qi-consistent and tight, then X is negatively curved.*

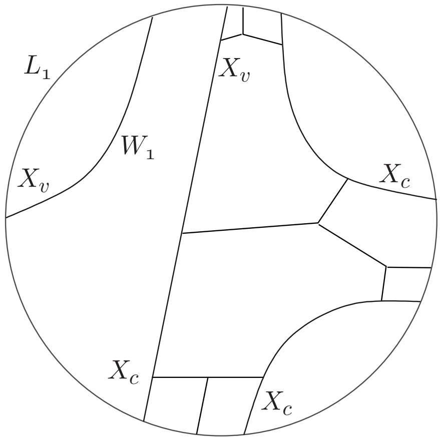

FIGURE 1. *Decomposition by walls of a disk Δ into polygons.*

We pass to the universal cover $\tilde{X}$. We would like to show that $\tilde{X}$ satisfies the subquadratic isoperimetric inequality, which would then imply that $\tilde{X}$ is a hyperbolic space [**2**, **6**]. We will use the techniques employed by Bestvina and Feighn in the proof of their combination theorem [**1**].

Let $\gamma : S^1 \longrightarrow \tilde{X}$ be a circuit that is transverse to and has nonempty intersection with $\cup\{\widetilde{X_e} : e \text{ is an edge of } T\}$. We may also assume that γ is contained in the 1-skeleton of $\tilde{X}$ (see [**4**]). Following [**1**], we talk of *good disks*. There is a disk $\Delta : D^2 \longrightarrow \tilde{X}$ with boundary γ. The set $\mathcal{W} = \Delta^{-1}(\cup\{\widetilde{X_e} : e \text{ is an edge of } T\})$ divides D^2 into regions that are mapped into negatively curved vertex spaces; see Figure 1. Elements of $\mathcal{W}$ are called *walls*. We may assume that Δ has the following properties.

(1) The set $\mathcal{W}$ consists of properly embedded arcs in D^2.

(2) The length of $\Delta(\cup\mathcal{W})$ in $\tilde{X}$ is minimal over all disks satisfying condition (1).

(3) The closures of the components of $\Delta(D^2 \setminus (\cup\mathcal{W}))$ have areas bounded by A times the length of their boundaries, where A is a constant.

(4) Define $\mathcal{L}$ to be the set of closures of the components of $S^1 \setminus (S^1 \cap \cup\mathcal{W})$. We may assume that γ restricted to each element of $\mathcal{L}$ is a geodesic in the appropriate $\widetilde{X_v}$. We view γ as a polygon whose sides are elements of $\mathcal{L}$. Hence the number of sides of γ can be no more than $\ell(\gamma)$. (The length of each side is at least 1.)

A disk is good if it satisfies conditions (1)–(4).

Our goal is to bound the area of Δ by a subquadratic function of the length of its boundary. As noted earlier, this will imply the hyperbolicity of $\tilde{X}$, by [**2**, **6**]. For a good disk Δ, by property (3), we have

$$\mathrm{Area}(\Delta) \leqslant A\,(2\ell(\Delta(\cup\mathcal{W})) + \ell(\Delta(\cup\mathcal{L}))) \leqslant A\,(2\ell(\Delta(\cup\mathcal{W})) + \ell(\gamma)).$$

Therefore we need to bound $\ell(\Delta(\cup\mathcal{W}))$ in terms of $\ell(\gamma)$.

Let us denote by $\mathcal{P}$ the set of closures of the components of $D^2 \setminus \cup\mathcal{W}$, and let P be an element of $\mathcal{P}$. If $\Delta(P)$ is contained in a good vertex space $\widetilde{X_v}$, then the map Δ restricted to ∂P is a τ-quasigeodesic polygon in $\widetilde{X_v}$. Note that each wall $W \in \mathcal{W}$ is a side of at least one polygon $P \in \mathcal{P}$ for which $\Delta(P)$ is contained in a good vertex space, and so we need only to consider such polygons. In order to avoid cumbersome notation in what follows, we will write $\ell(W)$ and P when we really mean $\ell(\Delta(W))$ and $\Delta(P)$.

Lemma 2.4. *We have*

$$\sum\{\ell(W) : W \subset \text{a bigon } P \subset X_v\} \leqslant \tau\ell(\gamma).$$

Proof. If P is a bigon, one of whose sides is a wall W, then the other side s of that bigon is an element of $\mathcal{L}$. According to property (4) in the definition of good disks, the images of the elements of $\mathcal{L}$ under Δ are geodesics in appropriate vertex spaces, and hence $\ell(W) \leqslant \tau\ell(s)$. The lemma follows. □

Lemma 2.5. *We have*

$$\sum\{\ell(W) : W \subset \text{an } m\text{-gon } P,\ m \geqslant 4\} = O(\ell(\gamma)\log(\ell(\gamma))).$$

Proof. The polygon P is a τ-quasigeodesic polygon, and can be resolved using Lemma 2.1. We call a point $w \in W$ a *singular point* if it lies in a singular fiber of the resolution of the polygon P. These points will decompose W into the union of closed segments V.

Let $\mathcal{V}(W)$ denote the set of all such segments V, and let $\mathcal{V} = \cup\{\mathcal{V}(W) : W \in \mathcal{W}\}$. Since

$$\sum\{\ell(W) : W \subset \text{an } m\text{-gon } P,\ m \geqslant 4\} = \sum\{\ell(V) : V \in \mathcal{V}\}$$

we need to bound $\ell(V)$, for all $V \in \mathcal{V}$.

Note that singular fibers decompose the polygon P into a union of quadrilaterals and triangles. The case of triangles is relatively easy to handle. One of the sides of such a triangle is some $V \in \mathcal{V}$, and another (call it S) is contained in S^1. Considering how the resolution was formed, we have $\ell(V) \leqslant C(\ell(S))$, where the function C is a linear function, from Proposition 2.1(v). Hence

$$\sum\{\ell(V) : V \in \mathcal{V}\} \leqslant D(\ell(\gamma)) + \sum\{\ell(V) : V \text{ is a side of a quadrilateral in } P\},$$

where D is a linear function. Let us consider the case of a quasigeodesic quadrilateral Q with sides V_1 and V_2 (contained in $\widetilde{X}_e$ and $\widetilde{X}_f$, respectively) that are joined by singular fibers of the resolution r of the polygon P. We may assume that V_1 is the shorter of the two. According to Proposition 2.1, the distance between the images under Δ of the two endpoints of a fiber is at most B(the number of sides of P) $\leqslant B(\ell(\gamma))$. Since Q is contained in a δ-hyperbolic space, it is not hard to show that as long as

$$\ell(V_1) > 2\tau B(\ell(\gamma)) + d, \qquad \text{where } d = d(\tau, \varepsilon, R),$$

we can find subsegments $S_1 \subset V_1$ and $S_2 \subset V_2$ such that

$$d_H(S_1, S_2) \leqslant 12\delta + 4R(\tau + 1) + 2\varepsilon.$$

If $\ell(S_1) > L$ (taking L from Remark 2.2, where the constant k is $12\delta+4R(\tau+1)+2\varepsilon$), then both S_1 and S_2 belong to $\widetilde{X}_e$. By doing surgery (see Figure 2) we can shorten the lengths of the walls in our disk, which contradicts the assumption that Δ was a good disk.

Thus, if

$$\ell(V) > 2\tau B(\ell(\gamma)) + d + L, \qquad \text{for some } V \in \mathcal{V},$$

then Δ is not a good disk.

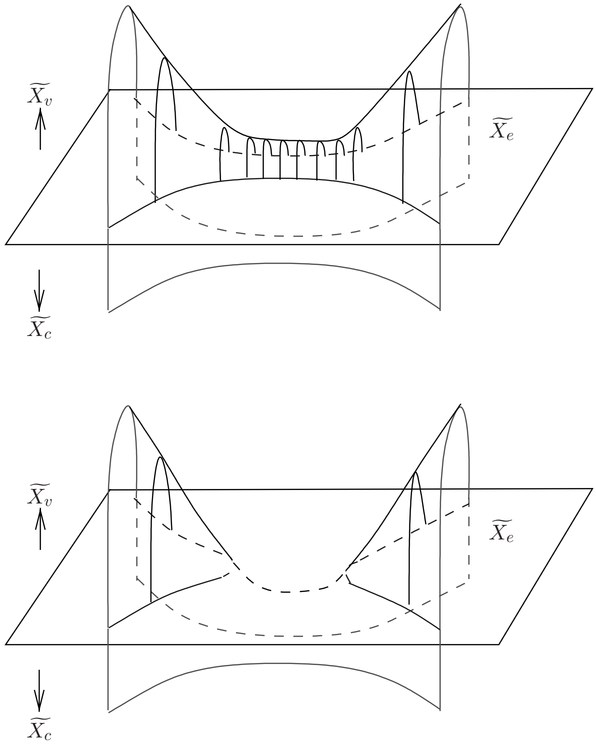

FIGURE 2. *Surgery: the lengths of the central 'parallel' curves are much shorter than the lengths of the walls connecting them. We change a disk by pushing this tunnel down into $\widetilde{X_c}$, and we obtain a new, good disk. For clarity, we have drawn $\widetilde{X_e}$ as two-dimensional.*

To complete the proof, we need to know the cardinality of $\mathcal{V}$. The number of segments $V \in \mathcal{V}$ is proportional to the number of singular fibers inside Δ. On the other hand, the number of singular fibers cannot be more than $\ell(\gamma) - 2$. Hence cardinality$(\mathcal{V}) = O(\ell(\gamma))$, and the lemma follows. $\square$

Proof of Theorem 2.3. If Δ is a good disk, then (as we have already noted)

$$\begin{aligned}\mathrm{Area}(\Delta) &\leqslant A(2\ell(\Delta(\cup\mathcal{W})) + \ell(\Delta(\cup\mathcal{L})))\\ &\leqslant A(2\ell(\Delta(\cup\mathcal{W})) + \ell(\gamma))\\ &\leqslant 2A(\tau\ell(\gamma) + O(\ell(\gamma)\log(\ell(\gamma)))) + A\ell(\gamma)\\ &= O(\ell(\gamma)\log(\ell(\gamma))).\end{aligned}$$

The last inequality follows from Lemmas 2.4 and 2.5. Therefore, $\tilde{X}$ satisfies the subquadratic isoperimetric inequality, and our proof is complete. $\square$

3. *Application to limit groups*

The goal of this section is to produce a δ-hyperbolic space on which a given limit group acts freely, by isometries. We will in fact consider a slightly larger class of groups, $\mathcal{C}$. We describe elements of $\mathcal{C}$ ($\mathcal{C}$-groups, for short) inductively.

Definition 3.1. A *torsion-free finitely generated group* G is a depth-0 $\mathcal{C}$-group if it is a finitely generated free group, or a finitely generated free abelian group, or the fundamental group of a closed hyperbolic surface. A torsion-free finitely generated group G is a $\mathcal{C}$-group of depth at most n if it has a graph of group decompositions with three types of vertices (abelian vertices, surface vertices, or vertices of depth at most $(n-1)$) and cyclic edge stabilizers, and if the following statements hold.

(1) Every edge is adjacent to at most one abelian vertex v. Further, G_v, the stabilizer of v, is a maximal abelian subgroup of G.

(2) Each surface vertex group is the fundamental group of a surface with boundary, and to each boundary component there corresponds an edge of this decomposition. Each edge group is conjugate to a boundary component.

(3) The stabilizer, G_v, of a vertex v of depth at most $(n-1)$, is a $\mathcal{C}$-group of depth at most $(n-1)$. The images in G_v of incident edge groups are distinct maximal abelian subgroups of G_v (that is, cyclic subgroups generated by distinct, primitive, hyperbolic elements of G_v).

We say that the *depth* of a C-group G is the smallest n for which G is of depth at most n.

We will obtain a hyperbolic space on which a $\mathcal{C}$-group G acts freely by induction on its depth. If G is a depth-0 $\mathcal{C}$-group, we take a tree, a horoball or $\mathbb{H}^2$.

Let G be a depth-n $\mathcal{C}$-group. For the vertex groups in the decomposition of G as above, we have the desired spaces by induction. Let $\tilde{X}/G$ be a graph of spaces corresponding to this splitting, and let $\tilde{X}$ be its universal cover. Let T_G be the corresponding graph of groups, and let T be a tree such that $T/G = T_G$. Our goal is to show that $\tilde{X}/G$ satisfies the hypotheses of Theorem 2.3, and consequently that $\tilde{X}/G$ is a hyperbolic space.

The requirement that we imposed on the splitting of G guarantees that X/G satisfies both the partial qi-embedded condition and the qi-consistency condition. That is, the generators of all edge groups adjacent to a nonabelian vertex group are identified with hyperbolic elements of that vertex group, and hence the corresponding edge spaces in $\tilde{X}$ are glued along quasigeodesics in the relevant vertex space. We note that no edge space qi-embeds into a vertex space that is a horoball.

Lemma 3.2. *$\tilde{X}/G$ is a tight graph of spaces.*

Proof. Suppose that the edge spaces $\widetilde{X_e}$ and $\widetilde{X_f}$ qi-embed into the vertex space $\widetilde{X_v}$. We noted that they are glued along quasigeodesics, say c_1 and c_2 respectively. There are elements g_1 and g_2 of $\pi_1 X_v$ that act as translations along c_1 and c_2, respectively. If the Hausdorff distance between the quasigeodesics c_1 and c_2 is bounded, we conclude that g_1 and g_2 fix the same two points in $\partial\widetilde{X_v}$. Hence they are both contained in a unique elementary group that is virtually cyclic. Since we have no torsion elements, this elementary group is cyclic, contradicting the choice of the splitting for G. $\square$

The definition of *relatively hyperbolic groups* appears in many forms; we use that given by Gromov in [**6**, 8.6].

Let X be a complete hyperbolic locally compact geodesic space with a discrete free isometric action of a group Γ such that the quotient $V = X/\Gamma$ is quasiisometric to the union of k copies of $[0,\infty)$, joined at 0. Lift k rays that correspond to ∂V to rays $r_i : [0,\infty) \longrightarrow X$, $i = 1,\ldots,k$. Let h_i be the horofunction corresponding to r_i, and let $r_i(\infty)$ be the limit point of r_i. Denote by $\Gamma_i < \Gamma$ the stabilizer of $r_i(\infty)$, and assume that it preserves h_i. Denote by $B_i(\rho)$ the horoballs $h_i^{-1}(-\infty,\rho) \subset X$, and assume that for sufficiently small ρ the intersections $\gamma B_i(\rho) \cap B_j(\rho)$ are empty unless $i = j$ and $\gamma \in \Gamma_i$. Let

$$\Gamma B(\rho) = \bigcup_{i,\gamma} \gamma B_i(\rho), \qquad i = 1,\ldots,k,\ \gamma \in \Gamma.$$

Let $X(\rho) = X \setminus \Gamma B(\rho)$, and assume that $X(\rho)/\Gamma$ is compact for all $\rho \in (-\infty,\infty)$.

Definition 3.3. We say that a group Γ is *hyperbolic relative to the subgroups* $\Gamma_1,\ldots,\Gamma_k$ if Γ admits an action on some X as above, and where Γ_i are the stabilizers of h_i.

After inspection of the action of $G \in \mathcal{C}$ on the δ-hyperbolic space $\tilde{X}$ that we constructed above, we see that all the requirements of Definition 3.3 are satisfied. Hence we have proved the following theorem.

Theorem 3.4. *Groups in $\mathcal{C}$ are hyperbolic relative to the collection of the conjugacy classes of their maximal noncyclic abelian subgroups.*

Since limit groups belong to the class $\mathcal{C}$ (see [**10**, 3.2 and 4.1], one consequence of this theorem is the following corollary, which was also noted in the work of Dahmani [**5**].

Corollary 3.5. *A limit group L is hyperbolic relative to the collection of representatives of conjugacy classes of its maximal noncyclic abelian subgroups.*

Several nice properties follow from the relative hyperbolicity. In [**3**], Bumagin shows that the conjugacy problem is solvable for a group G that is hyperbolic relative to a subgroup H with the solvable conjugacy problem; hence the conjugacy problem is solvable for limit groups. Also, Rebecchi has shown in [**9**] that a group G that is hyperbolic relative to a biautomatic subgroup H is itself biautomatic. We therefore conclude that limit groups are biautomatic.

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